Calculation of Multiconductor Microstrip Line Capacitances using the Semidiscrete Finite Element Method

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Abstract—**In the presented analysis partial finite element discretization of the Poisson's equation is implemented. The governing partial differential equation is thus reduced to a coupled set of ordinary differential equations, which is solved analytically. Formulation of the solution with this technique is more general and versatile than with the method of lines. The method of lines is derived as a special case of the semidiscrete finite element method.**

Introduction

THE EFFECTIVENESS of the method of lines (MOL) in solving microstrip problems has been amply demonstrated by Pregla and his co-workers [1]. However, their technique is based on the finite-difference method, and as such has several inherent disadvantages. Among them are the need for special considerations in implementing boundary conditions, transition conditions at dielectric boundaries and edge conditions, as well as inaccurate discretization of general curved boundaries. Moreover, in three-dimensional (3-D) problems the rectangular grids used in the MOL discretization preclude truly *local* refinement of the mesh. It is well known that these problems can be overcome in the framework of the finite element method (FEM). Therefore it may be useful to examine the feasibility of a technique based on the partial discretization approach, as is the MOL, in the context of the FEM. Such an approach is in fact possible and will be examined here briefly. It is termed the semidiscrete finite element method (SD-FEM), to distinguish it from the standard, fully-discrete FEM.

As an example to demonstrate the technique, a two-dimensional static problem for a microstrip line is formulated and solved with the semidiscrete FEM. The method of lines formulation for the same problem is outlined as a special case of the SD-FEM.

II. Formulation

Consider a shielded multistrip transmission line with a stratified dielectric substrate. The cross-section is shown in Fig. 1. In order to evaluate the capacitances of the line shown in Fig. 1 it is necessary to solve a boundary value problem governed by the Poisson's equation

$$\nabla^2\phi(x,z) = -\frac{1}{\epsilon_n}\rho(x,z). \quad (1)$$

In the preceeding equation $\phi(x, z)$ denotes the potential, ϵ_n is the permittivity of the nth layer, and $\rho(x, z)$ represents the charge density distribution, defined as $\rho(x, z) = \sigma(x)\delta(z - z')$ if $n = 1$, and $\rho(x, z) = 0$ if $n = 2$. The boundary conditions for the potential can be stated as follows:

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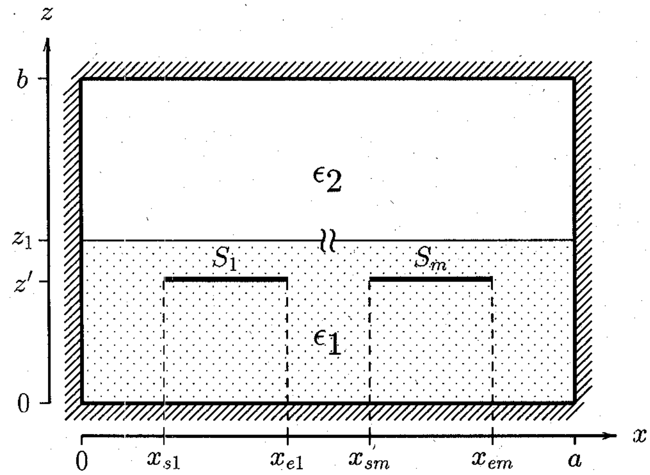

Fig. 1. Shielded microstrip lines in stratified dielectric medium.

$$\phi(x,z) = 0 \qquad \text{at } x = 0, a; \qquad z = 0, b \quad (2)$$

$$\phi(x,z) = \phi_i \qquad \text{if } (x,z) \in S_i, \qquad i = 1, m. \quad (3)$$

Additional transition conditions must be enforced at the dielectric interface ($z = z_1$). Solution of the posed problem yields $\sigma(x)$—the charge density distribution on the strips. The capacitance matrix elements can be then calculated using the standard definition.

The development of SD-FEM solutions is based on a *weak* statement of the problem. An appropriate *weak* formulation of (1) requires $\phi(x, z)$ to satisfy

$$\int_0^a \left[-\frac{\partial\phi}{\partial x}\frac{d\psi}{dx} + \frac{\partial^2\phi}{\partial z^2}\psi + \frac{1}{\epsilon_n}\rho\psi \right] dx + \frac{\partial\phi}{\partial x}\psi\Big|_{x=0}^{x=a} = 0 \quad (4)$$

for all values of z and all weighting functions $\psi(x)$. The latter are selected from a space of functions satisfying certain regularity conditions [2].

III. Finite Element Approximation

A combination of numerical and analytical methods is utilized to obtain a solution to (4). Initially, finite element techniques are used to approximate the solution variation with x. The x-domain, i.e., the interval $(0, a)$, is divided into a number of subintervals or elements. This subdivision, together with certain points—nodes—within each element, is known as the finite

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element mesh. A set of low-order polynomial basis functions, each spanning a small number of adjacent elements is generated. This basis set is used in the application of the Galerkin method to the formulation in (4). The sought solution is expanded (interpolated) in terms of the constructed basis set $\{\phi_i(x)\}_{i=1}^{N}$ as follows

$$\phi(x, z) = \sum_j v_j(z)\phi_j(x) = \tilde{\boldsymbol{\phi}}(x)\boldsymbol{v}(z), \tag{5}$$

where $v_j(z)$ is the value of the approximate solutions at the jth mesh node x_j, and $\phi_j(x)$ denotes the jth basis function. The boldfaced letters represent matrix quantities and tilde denotes transposition. As required in the Galerkin approach, representation (5) is substituted into the weak statement (4) and the weight functions $\psi = \phi_i$, $i = 1, N$ are used to test the resulting equation over the interval (0, a).

The outlined procedure yields the following system of N coupled, ordinary differential equations in the N unknowns $v_j(z)$

$$\boldsymbol{B}\frac{d^2\boldsymbol{v}(z)}{dz^2} - \boldsymbol{A}\boldsymbol{v}(z) = -\frac{1}{\epsilon_n}\boldsymbol{s}\delta(z - z'), \tag{6}$$

where $\boldsymbol{v}, \boldsymbol{s}$ are $N \times 1$ vectors and $\boldsymbol{A}, \boldsymbol{B}$ are sparse (banded), symmetric, positive-definite $N \times N$ matrices, whose elements are defined by

$$a_{ij} = \int_0^a \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx; \qquad b_{ij} = \int_0^a \phi_i\phi_j\,dx;$$

$$s_i = \int_0^a \sigma\phi_i\,dx. \tag{7}$$

Note that the boundary term in (4) does not contribute to the expressions in (7). This has been achieved by requiring that all the weight functions be zero at $x = 0, a$. Thus the boundary condition on the side walls is satisfied. If the Neumann conditions are stated, they can be incorporated by direct substitution into the boundary term in (4). Since the integrals in (7) can be easily evaluated in closed form, the CPU time required to fill the matrices is negligible.

The differential equation set (6) can be solved analytically, after the unknowns are decoupled by a linear transformation of the solution $\boldsymbol{v}$ to the *principal axis*, as proposed in [1]. A very expedient approach for solving the decoupled differential equations uses an analogy between the potential and its derivative with respect to z on the one hand, and voltage and current on a distributed transmission line on the other.

The boundary conditions at $z = 0, b$, as well as the transition conditions at dielectric interfaces, are applied in the process of solving for $\boldsymbol{v}(z)$. The remaining boundary conditions on the strips S_i are enforced last. This is accomplished by matching the elements of $\boldsymbol{v}(z)$ corresponding to nodes located on a particular strip to the prescribed potential on that strip, i.e., $v_j(z') = \phi_i$ if $x_j \in S_i$. The result of this procedure is a matrix equation for $\boldsymbol{s}$—the elements of which are moments of the charge density distribution $\sigma(x)$ with the weighting functions. After solving for $\boldsymbol{s}$, the total charge Q_i on the ith strip is determined simply by summing the elements of $\boldsymbol{s}$ corresponding to the given strip. A more detailed derivation of the outlined solution, applied to a 3-D problem, can be found in [3].

IV. Relationship with the Method of Lines

As mentioned in the Introduction, the MOL can be developed as a special case in the framework of the SD-FEM. This merely requires that the Poisson's equation be recast in a slightly different form, and a specific choice of basis functions. The MOL analysis of the problem considered above has been carried-out by Diestel [4]. His formulation can be reproduced using the SD-FEM approach.

Let Poisson's equation (1) be recast in the following form:

$$\frac{\partial\phi(x, z)}{\partial x} - \omega(x, z) = 0, \tag{8}$$

$$\frac{\partial\omega(x, z)}{\partial x} + \frac{\partial^2\phi(x, z)}{\partial z^2} + \frac{\rho(x, z)}{\epsilon_n} = 0. \tag{9}$$

An appropriate weighted-residual statement for the MOL derivation is obtained by using $\psi(x)$ and $\bar{\psi}(x)$ to test (8) and (9), respectively, on the interval (0, a). The suggested procedure yields

$$\int_0^a \left[\frac{\partial\phi(x, z)}{\partial x} - \omega(x, z)\right]\psi(x) = 0, \tag{10}$$

$$\int_0^a \left[\frac{\partial\omega(x, z)}{\partial x} + \frac{\partial^2\phi(x, z)}{\partial z^2} + \frac{\rho(x, z)}{\epsilon_n}\right]\bar{\psi}(x) = 0. \tag{11}$$

The MOL formulation follows from the weak statement of (10) and (11) if the following basis-testing set of functions is selected:

$$\phi_i(x), \bar{\psi}_i(x) = P(e_i, x - x_i), \qquad i = 1, N, \tag{12}$$

$$\omega_j(x), \psi_j(x) = P(h_j, x - x_j'), \qquad i = 1, N, \tag{13}$$

where $P(w, x - x_k)$ is the rectangular pulse function of width w, centered at $x = x_k$. The use of two separate sets of basis-testing functions in (10) and (11) to approximate the potential and its derivative, is analogous to using two different systems of *lines* in the context of MOL to evaluate the potential and its derivative. Moreover, because the derivative of a pulse function yields two impulse functions, the two sets of basis-testing functions have to shifted and *interlaced* with respect to each other to ensure well-posed integrals in (10) and (11). This shift also facilitates the application of boundary conditions [1]. An example demonstrating the positioning of the two basis sets for a problem involving both Dirichlet and Neumann boundary conditions is shown in Fig. 2.

V. Numerical Results

To test the validity and accuracy of the proposed technique, numerical experiments were performed employing a piecewise-linear basis set. The distribution of the nodes in the interval (0, a) was determined using the formula stated by Diestel [4]. This particular nodal distribution happens to equidistribute the error inherent in the linear interpolation of the solution, thus minimizing the overall error. It would not, however, be optimal when used with higher order polynomial basis functions. The calculated capacitance matrix for a five-strip transmission line is compared with the data computed by Diestel with the MOL. The SD-FEM results are presented in Table I. The data were computed using a total of 114 nodes in the mesh. The comparison with MOL solution obtained with 114 *lines* reveals that the two sets agree very well. The discrepancy between the two is less

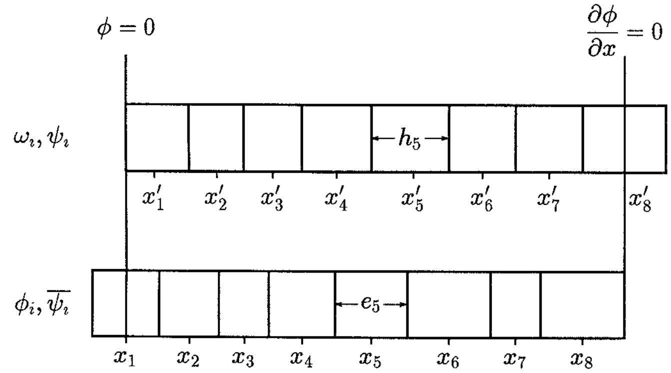

Fig. 2. Placement of basis functions of configuration for MOL formulation.

TABLE I
CAPACITANCE MATRIX FOR A 5-STRIP TRANSMISSION LINE

	Parameters
$a = 3.7,\ b \approx \infty$	$x_{s1} = 1.00\ x_{e1} = 1.16$
$z_1 = z' = 0.8$	$x_{s2} = 1.23\ x_{e2} = 1.70$
$\epsilon_{r1} = 2.5,\ \epsilon_{r2} = 1.0$	$x_{s3} = 1.77\ x_{e3} = 1.93$
	$x_{s4} = 2.00\ x_{e4} = 2.47$
	$x_{s5} = 2.54\ x_{e5} = 2.70$

Dimensions given in 'mm'

$$\frac{C}{\epsilon_0} = \begin{bmatrix} 4.683 & -2.589 & -0.080 & -0.064 & -0.013 \\ -2.589 & 7.934 & -2.377 & -0.553 & -0.064 \\ -0.080 & -2.377 & 5.802 & -2.377 & -0.080 \\ -0.064 & -0.553 & -2.377 & 7.934 & -2.589 \\ -0.013 & -0.064 & -0.080 & -2.589 & 4.683 \end{bmatrix}$$

than 1.6%. To illustrate the convergence properties of the SD-FEM with piecewise-linear basis functions, the relative error curves for two capacitance matrix elements are shown in Fig. 3. The error is determined by comparing the value of capacitance computed on a mesh containing N nodes, with the reference value obtained for the case where $N = 212$.

An additional numerical aspect of the solution that needs to be mentioned is the edge condition. Numerical experiments with SD-FEM, using the formulation (4), show that optimal convergence is obtained when the microstrip edge coincides with a node. Therefore the discretization process is somewhat simpler than in the MOL, where the *lines* adjacent to the edge must be placed at specific distances to satisfy the edge condition. The difference in behavior of the two solutions may stem from the fact that only the first derivative of the solution with respect to x

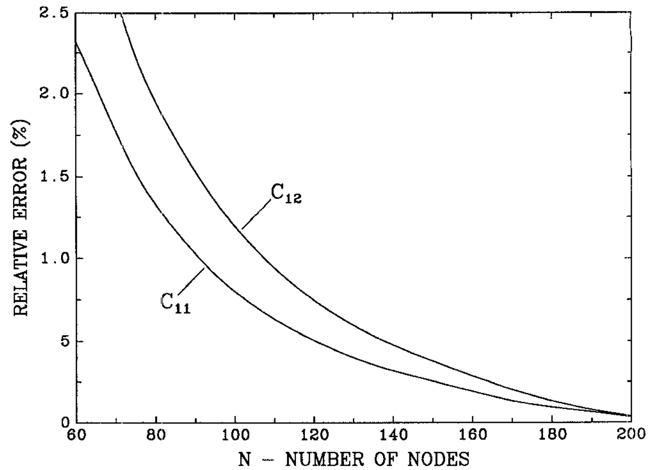

Fig. 3. Convergence curves for two elements of the capacitance matrix.

appears in the SD-FEM formulation, whereas in the MOL the second derivative needs to be approximated as well.

VI. CONCLUSION

The semidiscrete finite element method was applied to a static problem for microstrip lines. Although this method shares the method of lines' most important attribute, namely the partial discretization feature, it possesses different numerical characteristics. It is more general and versatile, capable of yielding MOL as a special case and eliminating the need for complicated meshes, while simplifying the application of boundary conditions. The advantages of this technique become even more apparent in 3-D problems [3].

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